FINITENESS OF A SECTION OF THE $SL(2,\mathbb{C})$-CHARACTER VARIETY OF KNOT GROUPS

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Abstract. We show that for any knot there exist only finitely many irreducible metabelian characters in the $SL(2,\mathbb{C})$-character variety of the knot group, and the number is given explicitly by using the determinant of the knot. Then it turns out that for any 2-bridge knot a section of the $SL(2,\mathbb{C})$-character variety consists entirely of all the metabelian characters, i.e., the irreducible metabelian characters and the single reducible (abelian) character. Moreover we find that the number of irreducible metabelian characters gives an upper bound of the maximal degree of the A-polynomial in terms of the variable l.

1. Introduction

Our main object in this paper is representations of the knot group into $SL(2,\mathbb{C})$. A representation ρ of a group G is said to be metabelian if the image $\rho([G,G])$ of the commutator subgroup $[G,G]$ is abelian. The irreducible metabelian representations are relatively easy to find in the irreducible representations in the following sense. Let us first denote the exterior of a knot K in 3-sphere S^3 by E_K, which is the complement of an open tubular neighborhood of K in S^3.

Proposition 1.1. *For an arbitrary knot K in S^3, any irreducible metabelian representation ρ of the knot group $\pi_1(E_K)$ into $SL(2,\mathbb{C})$ satisfies*

$$\mathrm{trace}(\rho(\mu)) = 0,\ \mathrm{trace}(\rho(\lambda)) = 2,$$

where (μ,λ) is the pair of the elements in $\pi_1(E_K)$ represented by the standard meridian and longitude of the knot K, respectively. In particular, ρ is not faithful.

This proposition shows that all the irreducible metabelian characters of the knot group $\pi_1(E_K)$ are on the section of the $SL(2,\mathbb{C})$-character variety cut by the equation $\chi_\rho(\mu) = \mathrm{trace}(\rho(\mu)) = 0$, where χ_ρ is the character of ρ.

In the paper [Li], X.-S. Lin shows the finiteness of the number of conjugacy classes of the irreducible metabelian representations of the knot group into $SU(2)$. Moreover he gives the number of the conjugacy classes by using the Alexander polynomial $\Delta_K(t)$. We can in fact generalize the result to the case of $SL(2,\mathbb{C})$.

Theorem 1.2. *For any knot K in S^3, there exist only finitely many conjugacy classes of the irreducible metabelian representations of the knot group $\pi_1(E_K)$ into $SL(2,\mathbb{C})$. Moreover the number of the conjugacy classes (i.e., the number of irreducible metabelian characters) is given by*

$$\frac{|\Delta_K(-1)| - 1}{2},$$

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where the absolute value $|\Delta_K(-1)|$ is so-called the determinant of K.

For example, the number of irreducible metabelian characters is $\frac{p-1}{2}$ for the 2-bridge knot $S(p,q)$ (the Schubert presentation) because $|\Delta_{S(p,q)}(-1)| = p$ (see Proposition 8 in Gaebler's thesis [G] for example). It turns out that the number of irreducible metabelian characters has an important meaning. For a finitely generated and finitely presented group G, let $X(G)$ be the $SL(2,\mathbb{C})$-character variety of G introduced in [CS]. Then for a knot K in S^3, we can define the A-polynomial $A_K(m,l) \in \mathbb{Z}[m,l]$ (refer to [CCGLS]). That is an invariant of knots constructed basically via the restriction

$$r : X(E_K) := X(\pi_1(E_K)) \to X(T^2) := X(\pi_1(T^2))$$

induced by the inclusion $i : \pi_1(T^2) \to \pi_1(E_K)$ (refer to [CCGLS] or Section 3).

Theorem 1.3. *For any 2-bridge knot $S(p,q)$, the number of irreducible metabelian characters in $X(\pi_1(E_{S(p,q)}))$ gives an upper bound of the maximal degree of the A-polynomial of $S(p,q)$ in terms of the variable l. In particular, the maximal degree is the number of irreducible metabelian characters surviving "discarding operations".*

We discuss the discarding operations at the very end of the proof of Theorem 1.3. Let $\deg_l(A_K(m,l))$ be the maximal degree of $A_K(m,l)$ in terms of l. Then the above theorem says that

$$\deg_l(A_{S(p,q)}(m,l)) \leq \frac{p-1}{2}.$$

This gives a representation theoretical proof of the upper bound of $\deg_l(A_{S(p,q)}(m,l))$ shown in Le's paper [Le].

As a by-product, we get criteria of irreducible non-metabelian representations. For example, we have the following.

Proposition 1.4. *If the A-polynomial of a knot K satisfies $A_K(\pm\sqrt{-1},l) = 0$, then there exist arcs in $X(E_K)$ each of which consists entirely of irreducible non-metabelian characters. If $A_K(m = \pm\sqrt{-1},l) \neq 0$ and has a factor other than $l-1$ or l and moreover the knot K is small, then there exists an irreducible non-metabelian representation satisfying* $\mathrm{trace}(\rho(\mu)) = 0$.

We remark that the determinant of knots no longer controls the A-polynomial for knots other than 2-bridge knots. For example, see the knot 8_{20}:

$$\deg_l(A_{8_{20}}(m,l)) = \deg_l(A_{8_{20}}(\pm\sqrt{-1},l)) = \deg_l((l-1)^3(l+1)^2) = 5,$$
$$\tfrac{|\Delta_{8_{20}}(-1)|-1}{2} = 4.$$

(See also Section 4.) In [N], we will introduce an algebraic variety via the Kauffman bracket skein module. The number of its irreducible components will give upper bounds of both quantities $\frac{|\Delta_K(-1)|-1}{2}$ and $\deg_l(A_K(m,l))$ with some conditions.

In this paper, we show the above statements in the following steps. In the next section, we give a proof of Theorem 1.2, which includes a proof of Proposition 1.1. In Section 3, we review the definition of the A-polynomial and give a proof of Theorem 1.3 and Proposition 1.4. Then the finiteness of the section of the $SL(2,\mathbb{C})$-character variety of the 2-bridge knot group is shown in the proof of Theorem 1.3. In the final section, we remark further aspects on this research.

2. Irreducible metabelian representations of knot groups

We first consider a convenient presentation of knot groups to research the irreducible metabelian representations.

Lemma 2.1 (Lemma 2.1 in [Li]). *The knot group $\pi_1(E_K)$ has a presentation*

$$\langle x_1, \cdots, x_{2g}, \mu \mid \mu\alpha_i\mu^{-1} = \beta_i, i = 1, \cdots, 2g\rangle,$$

where μ is represented by a meridian of K, α_i, β_i are certain words in $x_1, \cdots, x_{2g}$, and g is genus of a free Seifert surface of K.

This can be shown as follows (for more information refer to [Li]). Suppose S be a free Seifert surface of a knot K. Namely, $(N(S), \overline{S^3 - N(S)})$ is a Heegard splitting of S^3, where $N(S)$ is a closed tubular neighborhood of S in S^3. Let W_{2g} ($g = \text{genus}(S)$) be a spine of S, i.e., a bouquet (with a base point $*$) of circles in S which is a deformation retract of S. Denote the oriented circles in W_{2g} by $a_1, a_2, \cdots, a_{2g-1}, a_{2g}$ forming a symplectic basis of S with that order. Let $S\times[-1,1]$ be a bicollar of S given by the positive normal direction of S such that $S = S\times\{0\}$. Let $a_i^{\pm} = a_i \times \{\pm 1\}$ be circles in $W_{2g} \times \{\pm 1\}$ respectively, for $i = 1, \cdots, 2g$. Note that there exists a circle c on $\partial(S \times [-1, 1])$, uniquely determined up to isotopy, such that c connects two point $\{*\} \times \{\pm 1\}$ and its interior has no intersection with $W_{2g} \times \{\pm 1\}$. Consider the arc c as a base point and choose a basis $x_1, \cdots, x_{2g}$ for a free group $\pi_1(S^3 - S \times [-1, 1])$. Then, for $i = 1, \cdots, 2g$, we define α_i (resp. β_i) by the element of $\pi_1(S^3 - S \times [-1, 1])$, corresponding to a_i^+ (resp. a_i^-), which is a word in $x_1, \cdots, x_{2g}$. The circle $c \cup \{*\} \times [-1, 1]$ can be thought of as a meridian of a simple closed curve K' on S parallel to $K = \partial S$. Identifying K' with K and $\pi_1(S^3 - S)$ with $\pi_1(S^3 - S \times [-1, 1])$, we can show Lemma 2.1.

In the above setting, we have the following fundamental facts on α_i and β_i, $i = 1, \cdots, 2g$, due to [Li]. Let us denote by $v_{i,j}$ the exponent sum of x_j in α_i and $u_{i,j}$ the exponent sum of x_i in β_i. For two $2g \times 2g$ matrices $V := (v_{i,j})$ and $U := (u_{i,j})$ with integer entries, we have the followings:

- V is so-called the Seifert matrix of the Seifert surface S,
- $U = V^T$, where T means transpose.

By the definition of the Alexander polynomial, we have

$$|\det(V + V^T)| = |\Delta_K(-1)|. \tag{1}$$

Proof of Proposition 1.1 and Theorem 1.2: The proof is straightforward. It is well-known that there exist only two maximal abelian subgroups Hyp(=hyperbolic) and Para(=parabolic) in $SL(2,\mathbb{C})$, up to conjugation:

$$\text{Hyp} := \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL(2,\mathbb{C}) \,\middle|\, \lambda \in \mathbb{C}^* \right\}$$

$$\text{Para} := \left\{ \pm \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbb{C}) \,\middle|\, \omega \in \mathbb{C} \right\}$$

Note that two non-trivial elements g and h in $SL(2,\mathbb{C})$ are commutative if and only if the subgroup in $SL(2,\mathbb{C})$ generated by g and h is conjugate to a subgroup of Hyp

or Para. It is easy to see that the elements $x_1, \cdots, x_{2g}$ of $\pi_1(E_K)$ are in the commutator subgroup $[\pi_1(E_K), \pi_1(E_K)]$. Let ρ be an arbitrary irreducible metabelian representation. Then we can assume up to conjugation that

$$(\rho(x_i))_{1\leq i\leq 2g} = \left(\begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix} \right)_{1\leq i\leq 2g} \text{ or } \pm \left(\begin{pmatrix} 1 & \omega_i \\ 0 & 1 \end{pmatrix} \right)_{1\leq i\leq 2g},$$

where $\lambda_i(\neq 0)$ and ω_i are complex numbers. We can also check that $\mu x_i \mu^{-1}$, $i = 1, \cdots, 2g$, are in the commutator subgroup. With some linear algebra, we see that ρ is always an abelian representation if $\rho(x_i)$ are in Para. Hence we can ignore the parabolic case. In the hyperbolic case, we have

$$\rho(\mu) = \begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}, \ \rho(\mu x_i \mu^{-1}) = \begin{pmatrix} \lambda_i^{-1} & 0 \\ 0 & \lambda_i \end{pmatrix}, \text{ for } i = 1, \cdots, 2g,$$

where b is a non-zero complex number. Therefore, any irreducible metabelian representation satisfies $\mathrm{trace}(\rho(\mu)) = 0$. Note that the longitude λ can be described as a word in $x_1, \cdots, x_{2g}$ with zero exponent sum. Since x_i, $i = 1, \cdots, 2g$, are in the commutator subgroup, we have

$$\rho(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and thus $\mathrm{trace}(\rho(\lambda)) = 2$. In particular, ρ is not faithful. This gives a proof of Proposition 1.1.

Now, substituting the above equations to the defining relations of $\pi_1(E_K)$, we have the following equations for $\lambda_1, \cdots, \lambda_{2g}$:

$$\lambda_1^{w_{i,1}} \cdots \lambda_{2g}^{w_{i,2g}} = 1, \text{ for } i = 1, \cdots, 2g, \tag{2}$$

where $(w_{i,j}) = V + V^T$, the absolute value $|\det(w_{i,j})|$ is the determinant $|\Delta_K(-1)|$ of K (see Equation (1)). Put $\lambda_i := r_i e^{\sqrt{-1}\theta_i}$, $r_i > 0$, $0 \leq \theta_i < 2\pi$, for $i = 1, \cdots, 2g$, to solve Equations (2). Then we get

$$r_1^{w_{i,1}} \cdots r_{2g}^{w_{i,2g}} = 1, \tag{3}$$

$$e^{\sqrt{-1}(w_{i,1}\theta_1 + \cdots + w_{i,2g}\theta_{2g})} = 1. \tag{4}$$

We first solve Equations (3). Taking a function $\log = \log_e$ for its both sides, we have

$$(w_{i,j})(\log r_1, \cdots, \log r_{2g})^T = (0, \cdots, 0)^T,$$

where T is the transpose. Let us take the inverse matrix of $(w_{i,j})$ for both sides. Indeed, this can be successfully done because $|\det(w_{i,j})| = |\Delta_K(-1)| \neq 0$. Since the function log is bijective on the real numbers $\mathbb{R}$, we have only a single solution

$$(r_1, \cdots, r_{2g}) = (1, \cdots, 1).$$

We next solve Equations (4). They are equivalent to the following. Let us think of $\{\theta_i\}_{i=1,\cdots,2g}$ as elements $\{[\theta_i]\}_{i=1,\cdots,2g}$ of the quotient space $\mathbb{R}/(2\pi n \sim 2\pi m) \cong [0, 2\pi)$. Then we have

$$w_{i,1}[\theta_1] + \cdots + w_{i,2g}[\theta_{2g}] = [0], \text{ for } i = 1, \cdots, 2g.$$

They can be encoded into a single form by using the matrix $(w_{i,j})$ as follows:

$$(w_{i,j})([\theta_1], \cdots, [\theta_{2g}])^T = ([0], \cdots, [0])^T. \tag{5}$$

Then the fundamental fact on the order ideal gives the number of solutions of the above equations as the absolute value of the determinant of the matrix $(w_{i,j})$ (for example, see p.205 of [Ro]). Recalling $|\det(w_{i,j})| = |\Delta_K(-1)|$, we see that $|\Delta_K(-1)|$ is the number of solutions of Equations (4) and thus Equation (2).

Now, we discard the trivial solution $(\lambda_1, \cdots, \lambda_{2g}) = (1, \cdots, 1)$ from $|\Delta_K(-1)|$ solutions since it gives an abelian representation. We remark that a non-trivial solution of Equations (2) always gives an irreducible representation, because a representation ρ of a finitely generated and finitely presented group G into $SL(2,\mathbb{C})$ is reducible if and only if $\mathrm{trace}(\rho(g)) = 2$ for any $g \in [G,G]$ (see Corollary 1.2.1 of [CS]). Remember that there still exist infinitely many possibility for ρ to be an irreducible metabelian representation because of b in $\rho(\mu)$. However they are all conjugate. Namely, we first fix a non-trivial solution (λ_i). Let $\rho_{(\lambda_i),b}$ and $\rho_{(\lambda_i),b'}$ be representations associated with non-zero complex numbers b and b' for $\rho(\mu)$ along with the above fixed solution (λ_i), respectively. Then we can easily show with some linear algebra that they are always conjugate. Hence we do not care about b for the conjugacy classes. On the other hand, if $(\lambda_1, \cdots, \lambda_{2g})$ is a solution of Equations (4), then $(\lambda_1^{-1}, \cdots, \lambda_{2g}^{-1})$ is also a solution. That is, let $\rho_{(\lambda_i),b}$ and $\rho_{(\lambda'_i),b}$ be representations defined in the same fashion as above. Then it can be easily shown that they are conjugate if and only if $(\lambda'_i) = (\lambda_i^{-1})$ or (λ_i). Combining those facts on conjugation, we see that the number of conjugacy classes of the irreducible metabelian representations is $\frac{|\Delta_K(-1)|-1}{2}$. □

3. Representation theoretical meaning of the maximal degree of the A-polynomial in terms of l

We first review the A-polynomial of knots (for more information, refer to [CCGLS]). For a finitely generated and finitely presented group G, the $SL(2,\mathbb{C})$-character variety $X(G)$ is defined as the quotient of $\mathrm{Hom}(G, SL(2,G))$ by the trace function t_g, $g \in G$,

$$t_g : \mathrm{Hom}(G, SL(2,\mathbb{C})) \to \mathbb{C},\ t_g(\rho) := \mathrm{trace}(\rho(g)),$$

i.e., two representations ρ_1 and ρ_2 are equivalent if $t_g(\rho_1) = t_g(\rho_2)$ for any $g \in G$. Actually, it turns out that the set $X(G)$ can be identified with an algebraic variety in some space $\mathbb{C}^N$.

Now, for a knot K in S^3, we consider the restriction

$$r : X(E_K) := X(\pi_1(E_K)) \to X(\partial E_K = T^2) := X(\pi_1(T^2)),\ r(\chi_\rho) := \chi_{\rho\circ i}$$

induced by the inclusion $i : \pi_1(T^2) \to \pi_1(E_K)$. Every irreducible component of $X(E_K)$ has 1- or 0-dimensional closure of the image under r (refer to [CCGLS] and also Lemma 2.1 in [DG]). Let Δ be the set of pairs of diagonal matrices

$$\Delta := \left\{ \left(\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \right) \middle| (m,l) \in \mathbb{C}^* \times \mathbb{C}^* \right\}$$

Then take a preimage $\mathcal{E}(K)$ of $r(X(E_K))$ under a 2-fold branched covering map $p : \Delta \to X(T^2)$ defined by

$$p\left(\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \right) := (m + m^{-1}, l + l^{-1}, ml + m^{-1}l^{-1}) \in \mathbb{C}^3.$$

Identifying Δ with the set $\mathbb{C}^* \times \mathbb{C}^*$ via the correspondence

$$\left(\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}, \begin{pmatrix} l & 0 \\ 0 & l^{-1} \end{pmatrix} \right) \to (m,l) \in \mathbb{C}^* \times \mathbb{C}^*,$$

we can think of $\mathcal{E}(K)$ as a subset (also denoted by $\mathcal{E}(K)$) in $\mathbb{C}^* \times \mathbb{C}^*$. Taking the closure of $\mathcal{E}(K)$ in $\mathbb{C}^2$, we get an algebraic variety $\overline{\mathcal{E}(K)}$ in $\mathbb{C}^2$ (called the eigenvalue variety of knot K) each of whose irreducible components is 1- or 0-dimensional. Discard all the 0-dimensional components and denote the remains by D_K. The defining polynomial of D_K in $\mathbb{Z}[m,l]$, uniquely determined up to non-zero constant multiple, is called the A-polynomial of knot K and denoted by $A_K(m,l)$. Since $A_K(m,l)$ has always the factor $l-1$, which comes from the characters of abelian representations, we divide $A_K(m,l)$ by $l-1$ and again denote the resulting polynomial by $A_K(m,l)$. Note that the variable m of $A_K(m,l)$ has only even power.

The set $\overline{\mathcal{E}(K)} - \mathcal{E}(K)$ may have finitely many points. Such a point, say (m_0, l_0), is called a hole of the eigenvalue variety $\overline{\mathcal{E}(K)}$ if that satisfies $m_0 \neq 0$ and $l_0 \neq 0$. We remark that if $\overline{\mathcal{E}(K)}$ has a hole, then the knot exterior has a closed essential surface (see Section 5 in [CL] for example).

Proof of Theorem 1.3: Let us first recall the 2-bridge knots. For coprime odd integers p and q, $(p > 0,\ p > |q| > 0)$, there exists associated the 2-bridge knot $S(p,q)$ (refer to [K] for example). The knot group $\Gamma_{p,q} := \pi_1(E_{S(p,q)})$ has the form

$$\Gamma_{p,q} = \langle x_1, x_2 \mid wx_1 = x_2 w,\ w := x_1^{e_1} x_2^{e_2} \cdots x_2^{e_{p-1}} \rangle,$$

where $e_i := (-1)^{[\frac{iq}{p}]}$ for $0 \leq i \leq p-1$, $[\cdot]$ is the Gaussian integer. By the definition of the A-polynomial, it suffices to focus on the non-abelian representations to prove the theorem. It follows from Lemma 1 in [Ri] that any non-abelian representation $\rho : \Gamma_{p,q} \to SL(2,\mathbb{C})$ can be conjugated so that

$$\rho(x_1) = \begin{pmatrix} t^{1/2} & t^{-1/2} \\ 0 & t^{-1/2} \end{pmatrix},\ \rho(x_2) = \begin{pmatrix} t^{1/2} & 0 \\ -t^{1/2}u & t^{-1/2} \end{pmatrix}.$$

Put

$$\rho(w) := \begin{pmatrix} w_{11}(t,u) & w_{12}(t,u) \\ w_{21}(t,u) & w_{22}(t,u) \end{pmatrix},\ w_{ij}(t,u) \in \mathbb{Z}[t, t^{-1}, u].$$

Then Theorem 1 in [Ri] combined with some easy calculation shows that ρ is a representation of $\Gamma_{p,q}$ into $SL(2,\mathbb{C})$ if and only if t and u satisfy

$$w_{11}(t,u) + (1-t) w_{12}(t,u) = 0. \tag{6}$$

We now focus on the section $S_0(S(p,q))$ of $X(E_{S(p,q)})$ using the equation $\chi_\rho(\mu) = \mathrm{trace}(\rho(\mu)) = 0$, where χ_ρ is the character of ρ. This means assuming $t = -1$, and thus

$$\rho(x_1) = \begin{pmatrix} \sqrt{-1} & -\sqrt{-1} \\ 0 & -\sqrt{-1} \end{pmatrix},\ \rho(x_2) = \begin{pmatrix} \sqrt{-1} & 0 \\ -\sqrt{-1}u & -\sqrt{-1} \end{pmatrix}.$$

Then Equation (6) goes to

$$w_{11}(-1,u) + 2 w_{12}(-1,u) = 0. \tag{7}$$

Since $\rho(x_i)^{-1} = -\rho(x_i)$ at $t = -1$ for $i = 1, 2$, and $e_j = e_{(p-1)-j}$ for $1 \leq j \leq p-1$, we have

$$\rho(w) = (\rho(x_1)\rho(x_2))^{\frac{p-1}{2}} = \begin{pmatrix} -1-u & -1 \\ -u & -1 \end{pmatrix}^{\frac{p-1}{2}}.$$

By induction, we get $\deg_u(w_{11}(-1,u)) = \frac{p-1}{2}$, $\deg_u((w_{12})(-1,u)) = \frac{p-3}{2}$. Hence Equation (7) have $\frac{p-1}{2}$ solutions over $\mathbb{C}$ with multiplicity. Remember that there exist $\frac{|\Delta_{S(p,q)}(-1)|-1}{2} = \frac{p-1}{2}$ characters of irreducible metabelian representations on the section $S_0(S(p,q))$. So all $\frac{p-1}{2}$ solutions of Equation (7) must be distinct and associated representations are irreducible metabelian and become $\frac{p-1}{2}$ distinct points on $S_0(S(p,q))$. Namely, $S_0(S(p,q))$ consists of the abelian character and $\frac{p-1}{2}$ points corresponding to the irreducible metabelian characters. Note that for any knot K, $S_0(K)$ has only irreducible characters except the abelian character, since $\Delta_K(-1) \neq 0$ (refer to [HPP] or the original papers [B, dR]).

Now, for any 2-bridge knot the eigenvalue variety has no holes, because any 2-bridge knot is small, namely the knot exterior has no closed essential surfaces. Moreover Proposition 2 in the paper of Hatcher and Thurston [HT] shows that $S(p,q)$ has no meridional boundary slope. Then it follows from Theorem 3.4 in [CCGLS] that there exist no vertical edges in the Newton polygon of the A-polynomial of $S(p,q)$. Hence $A_{S(p,q)}(\sqrt{-1}, l)$ must have constant term and thus $(m,l) = (\sqrt{-1}, 0)$ is not a solution of $A_{S(p,q)}(\sqrt{-1}, l) = 0$. Therefore every solution of $A_{S(p,q)}(\sqrt{-1}, l) = 0$ comes from $S_0(S(p,q))$. This means that the number of solutions of $A_{S(p,q)}(\sqrt{-1}, l) = 0$ with multiplicity, which is equal to the maximal degree of $A_{S(p,q)}(\sqrt{-1}, l)$, must be less than or equal to the number of irreducible metabelian characters in $S_0(S(p,q))$. Note that the above inequality can be caused by the fact that we do not consider the 0-dimensional components of D_K and we define the A-polynomial so that the polynomial has no repeated factors.

Now recall that the presentation of longitude λ using the generators x_1, x_2 is

$$\lambda = \omega^{-1} \cdot \widetilde{\omega} \cdot x_1^{2\sigma},$$

where $\widetilde{\omega} := x_1^{-\varepsilon_1} x_2^{-\varepsilon_2} \cdots x_2^{-\varepsilon_{p-1}}$, $\sigma := \sum_{i=1}^{p-1} \varepsilon_i$. Since $\rho(x_i)^{-1} = -\rho(x_i)$ for $i = 1, 2$, we have $\rho(\lambda) = \mathrm{id}$. So the A-polynomial at $m = \sqrt{-1}$ has the factorization

$$A_{S(p,q)}(\sqrt{-1}, l) = (l-1)^{k_{p,q}}, \ k_{p,q} \in \mathbb{Z}_{\geq 0}.$$

This shows that

$$\frac{p-1}{2} \geq k_{p,q} = \deg_l(A_{S(p,q)}(\sqrt{-1}, l)).$$

Since there exist no vertical edges in the Newton polygon of $A_{S(p,q)}(m,l)$, we have

$$\deg_l(A_{S(p,q)}(\sqrt{-1}, l)) = \deg_l(A_{S(p,q)}(m,l)).$$

Therefore we get

$$\frac{p-1}{2} \geq k_{p,q} = \deg_l(A_{S(p,q)}(\sqrt{-1}, l)) = \deg_l(A_{S(p,q)}(m,l)).$$

This shows the first statement in Theorem 1.3.

As regards the meaning of the maximal degree of the A-polynomial of $S(p,q)$, the process of throwing away the 0-dimensional components of D_K can make a difference

between $\frac{p-1}{2}$ and $\deg_l(A_{S(p,q)}(m,l))$. We also remark that the A-polynomial is defined so that there are no repeated factors in it, namely the multiplicity of the curve components in D_K are ignored in the construction process of the A-polynomial. This discarding process can also decrease the number of irreducible metabelian characters in the target. So we have to count the number of irreducible metabelian characters surviving the above *discarding operations* when calculating the exact maximal degree of $A_{S(p,q)}(m,l)$ in terms of l. Hence we can say that $\deg_l(A_{S(p,q)}(m,l))$ is the number of irreducible metabelian characters surviving the discarding operations. □

Proof of Proposition 1.4: Let us first factorize $A_K(\pm\sqrt{-1},l)$ over $\mathbb{C}$. If

$$A_K(\pm\sqrt{-1},l)=0,$$

then $A_K(m,l)$ must have a factor m^2+1, since m^2+1 is the minimal polynomial of $\sqrt{-1}$ over $\mathbb{Z}$. Then there exist two irreducible components $m=\pm\sqrt{-1}$ of D_K in $\mathbb{C}^2$, denoted respectively by $C_\pm$. By Proposition 1.2, all the points of $C_\pm$ except $(m,l)=(\pm\sqrt{-1},1)$ and holes correspond to characters of irreducible non-metabelian representations. Hence there must exist arcs in $X(E_K)$ associated with $C_\pm-\{(\pm\sqrt{-1},1)\}\cup\{\text{holes}\}$ consisting entirely of characters of irreducible non-metabelian representations. This completes the proof of the first criterion.

As regarding the second criterion, we assume that there exists a factor $l-\omega$, $\omega(\neq 0,1)\in\mathbb{C}^*$. Note that the points $(m,l)=(\pm\sqrt{-1},\omega)$ are not holes since K is small. It follows from Proposition 1.1 that the characters of all the irreducible metabelian representations surviving the discarding operations, as well as all the abelian characters, correspond to the factor $l-1$. Hence we see that there exists at least a character coming from an irreducible non-metabelian representation ρ surviving the discarding operations such that $\mathrm{trace}(\rho(\mu))=0$, $\mathrm{trace}(\rho(\lambda))=\omega+\omega^{-1}$. This completes the proof of the second criterion.

□

4. Remarks

As remarked in Section 1, the determinant of knots no longer controls the A-polynomial for knots other than 2-bridge knots. However it is possible that the quantity $\frac{|\Delta_K(-1)|-1}{2}$ will be able to give an upper bound of the multiplicity of the factor $l-1$ of $A_K(\pm\sqrt{-1},l)$ for any K. To check this, one may want to observe whether or not every character χ_ρ on the section $S_0(K)$ satisfying $\chi_\rho(\lambda)=2$ is a metabelian character.

As performed in Section 3, the fact that the Newton polygon of $A_{S(p,q)}(m,l)$ has no vertical edges as well as the fact that there exist no holes in $\overline{\mathcal{E}(S(p,q))}$ is very useful to study the A-polynomial by using the section $S_0(S(p,q))$. It will be very interesting to research whether or not there exist vertical edges in the Newton polygon of $A_K(m,l)$ and there exists a hole in $\overline{\mathcal{E}(K)}$ for knots other than 2-bridge knots.

In this paper, we have a little discussion on the exact degree of the A-polynomial of the 2-bridge knots by using the word "discarding operations". We will discuss more this topic at another time.

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References

[B] **G. Burde**, *Darstellungen von Knottengruppen*, Math. Ann. **173** (1967), 24–33.

[CCGLS] **D. Cooper, M. Culler, H. Gillet, D. Long and P. Shalen**, *Plane curves associated to character varieties of 3-manifolds*, Invent. Math. **118** (1994), 47–84.

[CL] **D. Cooper and D. Long**, *Remarks on the A-polynomial of a knot*, J. Knot Theory Ramifications **5** (1996), 609–628.

[CS] **M. Culler and P. Shalen**, *Varieties of group representations and splittings of 3-manifolds*, Ann. of Math. **117** (1983), 109–146.

[DG] **N. Dunfield and S. Garoufalidis**, *Non-triviality of the A-polynomial for knots in S^3*, Algebr. Geom. Topol. **4** (2004), 1145-1153.

[dR] **G. de Rham**, *Introduction aux polynômes d'un nœud*, Enseign. Math. **13** (1967), 187–194.

[G] **R. Gaebler**, *Alexander Polynomials of Two-Bridge Knots and Links*, available at `http://www.math.hmc.edu/seniorthesis/archives/2004/rgaebler/rgaebler-2004-thesis.pdf`.

[HPP] **M. Heusener, J. Porti and E. Peiró**, *Deformations of reducible representations of 3-manifolds into* $\mathrm{SL}_2(\mathbb{C})$, J. Reine Angew. Math. **530** (2001), 191–227.

[HT] **A. Hatcher and W. Thurston**, *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.

[K] **A. Kawauchi**: *A survey of knot theory*, Birkhäuser Verlag, Basel, 1996.

[Le] **T.T.Q. Le**, *The colored Jones polynomial and the A-polynomial of knots*, preprint.

[Li] **X.-S. Lin**, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin., Engl. **17** (2001), 361–380.

[N] **F. Nagasato**, *Algebraic varieties via a filtration of the Kauffman bracket skein module*, in preparation.

[Ri] **R. Riley**, *Non abelian representations of 2-bridge knot groups*, Quart. J. Math. Oxf. **35** (1984), 191–208.

[Ro] **D. Rolfsen**, *Knots and links*, Mathematics Lecture Series **7**, Publish or Perish, Berkeley, California, 1976.

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